Weierstrass Gap Sequence at Total Inflection Points of Nodal Plane Curves

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0 Introduction

Let Γ be a plane curve of degree d with δ ordinary nodes and no other singularities. Let C be the normalization of Γ. Let $g = \dfrac{(d-1)(d-2)}{2} - \delta$; the genus of C. We identify smooth points of Γ with the corresponding points on C. In particular, if P is a smooth point on Γ then the Weierstrass gap sequence at P is considered with respect to C. A smooth point $P \in \Gamma$ is called an $(e-2)$-inflection point if $i(\Gamma, T; P) = e \geq 3$ where T is the tangent line to Γ at P (cf. Brieskorn–Knörrer[1, p. 372]). Of course, $e \leq d$ and a 1-inflection point is an ordinary flex. In particular, a $(d-2)$-inflection point is called a total inflection point.

Let N be the semigroup consisting of the non-gaps of P, so $\mathbf{N} - N = \{\alpha_1 < \alpha_2 < \cdots < \alpha_g\}$ is the Weierstrass gap sequence of P. Clearly $\{d-1, d\} \subset N$, so $N_d := \{a(d-1) + bd | a, b \in \mathbf{N}\} \subset N$ (see also Lemma 1.2).

Let $k = \min\{\ell \in \mathbf{N} | \delta \leq \dfrac{\ell(\ell+3)}{2}\}$ and let

$$N^{(1)}_{d,\delta} = N_d \cup \{n \in \mathbf{N} | n \geq (d-k-3)d + \frac{k(k+3)}{2} - \delta + 2\}.$$

Let $\mathbf{N} - N^{(1)}_{d,\delta} = \{\alpha^{(1)}_1 < \alpha^{(1)}_2 < \cdots < \alpha^{(1)}_g\}$. One has $\alpha_i \geq \alpha^{(1)}_i$ for $1 \leq i \leq g$. So $N^{(1)}_{d,\delta}$ is the minimal (in the sense of weight) possible semigroup of non-gaps.

For $\delta \in \{0, 1\}$, one has $N = N^{(1)}_{d,\delta}$. For $\delta \geq 2$ there exist pairs of $(\Gamma; P)$ as above with $N \neq N^{(1)}_{d,\delta}$. We give a list of all possible values for N in case $2 \leq \delta \leq 5$. (see end of §1).

Define $N_{d,1}^{(\max)} = N_{d,1}^{(\max 2)} = N_{d,1}$ and, by means of induction, for $\delta \geq 2$,

$$\begin{aligned} N_{d,\delta}^{(\max)} &= N_{d,\delta-1}^{(\max)} \cup \{(d-\delta-2)d+1\} \\ N_{d,\delta}^{(\max 2)} &= N_{d,\delta-1}^{(\max 2)} \cup \{(d-\delta-2)d+\delta\}. \end{aligned}$$

$N_{d,\delta}^{(\max)}$ (resp. $N_{d,\delta}^{(\max 2)}$) is a semigroup if and only if $d \geq 2\delta+1$ (resp. 2δ). Let

$$\begin{aligned} \mathbf{N} - N_{d,\delta}^{(\max)} &= \{\alpha_1^{(\max)} < \alpha_2^{(\max)} < \cdots < \alpha_g^{(\max)}\} \\ \mathbf{N} - N_{d,\delta}^{(\max 2)} &= \{\alpha_1^{(\max 2)} < \alpha_2^{(\max 2)} < \cdots < \alpha_g^{(\max 2)}\}. \end{aligned}$$

We prove that $\alpha_i \leq \alpha_i^{(\max)}$ for $1 \leq i \leq g$ and if $N \neq N_{d,\delta}^{(\max)}$, then $\alpha_i \leq \alpha_i^{(\max 2)}$ for $1 \leq i \leq g$ (Lemma 3.1). So $N_{d,\delta}^{(\max)}$ (resp. $N_{d,\delta}^{(\max 2)}$) is the maximal (resp. up to 1 maximal) semigroup of non-gaps.

Our main results are the following:

1. There exist pairs $(\Gamma; P)$ such that $N = N_{d,\delta}^{(1)}$ (2.2),
2. If $d \geq 2\delta+1$ (resp. $d \geq 2\delta$) then there exist pairs $(\Gamma; P)$ such that $N = N_{d,\delta}^{(max)}$ (resp. $N = N_{d,\delta}^{(max2)}$) (3.2).

The existence of Weierstrass points with gap sequence $\mathbf{N} - N_{d,\delta}^{(1)}$ is already proved in [4] for the case $\delta = \dfrac{d^2-7d+12}{2}$. The method used in that paper is completely different from ours. It has the advantage of not using plane models but the proof looks more complicated. It might be possible to prove our existence result in this way completely, but it might become very complicated. We didn't try it. Also, it gives an affirmative answer to Question 1 in [2] for the case $s = n+1$. It is not clear to us at the moment how to generalize the proof for the cases with $s \geq n+2$.

1 Generalities and low values for δ

To start, we deal with the case $\delta = 0$.

Lemma 1.1 *Let Γ be a smooth plane curve of degree d and let P be a total inflection point of Γ. Then $N_d = N_{d,0}^{(1)}$ is the semigroup of non-gaps of P.*

Proof. Let T be the tangent line at P, L_1 be a general line passing through P and let L_2 be a general line not passing through P. Then the curve $C(a,b) = aT + bL_1 + (d-3-a-b)L_2$ is canonical adjoint, if $0 \leq a \leq d-3, 0 \leq b \leq d-3-a$. Then we have $i(\Gamma.C(a,b);P) = ad+b$. Hence, $\{ad+b+1 : 0 \leq a \leq d-3, 0 \leq b \leq d-3-a\}$ is the gap sequence at P. This completes the proof.

In order to study the case $\delta > 0$, we prove some lemmas. For the rest of this section, Γ is a plane curve of degree d with $\delta(>0)$ ordinary nodes $s_1, \ldots, s_\delta$ as its only singularities. Also $P \in \Gamma$ is a total inflection point.

Lemma 1.2 *The set of nongaps at P contains $N_{d,0}$.*

Proof. Assume that $n \in N_{d,0}$. Let $\alpha = \left[\frac{n-1}{d}\right] + 1$, ℓ be the equation of T (the tangent line at P), ℓ_0 be the equation of a general line passing through P and let ℓ_1 be the equation of a general line. Considering

$$\frac{\ell_0^{\alpha d - n} \ell_1^{\alpha + n - \alpha d}}{\ell^\alpha},$$

we obtain that n is a nongap at P.

Lemma 1.3 *Let γ be a curve of degree less than d so that $i(\gamma, \Gamma; P) = k \geq d$. Then, T is a component of γ, i. e. there is a curve γ' of degree $\deg \gamma - 1$ such that $\gamma = \gamma' T$.*

Proof. Since $i(T, \Gamma; P) = d$ and $i(\gamma, \Gamma; P) \geq d$, by Namba's lemma [5, Lemma 2.3.2] (cf. Coppens and Kato [3, Lemma 1.1] for a generalization), we have $i(T, \gamma; P) \geq d > \deg \gamma$. Hence we have the desired result by Bezout's theorem.

By a successive use of this lemma we have:

Lemma 1.4 *Let γ be a canonical adjoint curve such that $i(\gamma, \mathit{\Gamma}; P) = \alpha d + \beta$ $(0 \leq \alpha \leq d-3, 0 \leq \beta \leq d-3-\alpha)$. Then, there is an adjoint curve γ' of degree $d-3-\alpha$ such that $\gamma = T^{\alpha}\gamma'$ and $i(\gamma', \mathit{\Gamma}; P) = \beta$.*

Using Lemma 1.4 we have the following corollaries:

Corollary 1.5 *If $\delta \geq 1$, then $i(\gamma, \mathit{\Gamma}; P) < (d-3)d$ for every canonical adjoint curve γ, hence $(d-3)d+1$ is a nongap at P.*

Corollary 1.6 *Assume that $\delta \geq 2$. Then, $(d-4)d+\beta+1$ $(\beta = 0$ or $1)$ is a gap if and only if there is a line L_0 such that $s_1, \ldots, s_\delta \in L_0$. Moreover, in this case, the following three conditions are equivalent:*

1. $P \notin L_0$, (resp. $P \in L_0$),
2. $(d-4)d+1$ (resp. $(d-4)d+2$) *is a gap,*
3. $(d-3-\alpha)d+1+\alpha$ $(\alpha = 1, \ldots, \delta-1)$ (resp. $(d-3-\alpha)d+1$ $(\alpha = 1, \ldots, \delta-1)$) *are nongaps.*

Proof. The existence of the line L_0 and the equivalence between (i) and (ii) follows immediately from Lemma 1.4.

Assume that $(d-4)d+1$ is a gap. If $(d-3-\alpha)d+\alpha+1$ $(1 \leq \alpha \leq \delta-1)$ is a gap then Lemma 1.4 provides an adjoint curve γ' of degree α with $i(\gamma', \mathit{\Gamma}; P) = \alpha$. So γ' has $s_1, \ldots, s_\delta$ as common points with L_0. Bezout's theorem implies that $\gamma' = \gamma'' L_0$ where γ'' is a curve of degree $\alpha - 1$ with $i(\gamma'', \mathit{\Gamma}; P) = \alpha$ (since $P \notin L_0$). Namba's lemma implies $\gamma'' = \gamma''' T$, but then $i(\gamma'', \mathit{\Gamma}; P) \geq d$, so $\delta \geq \alpha + 1 \geq d+1$. A contradiction since $s_1, \ldots, s_\delta$ are collinear.

Assume that $(d-4)d+2$ is a gap. If $(d-3-\alpha)d+1$ $(1 \leq \alpha \leq \delta-1)$ is a gap then Lemma 1.4 provides an adjoint curve γ' of degree α with $i(\gamma', \mathit{\Gamma}; P) = 0$. But $\gamma' = \gamma'' L_0$ and $P \in L_0$, hence a contradiction.

Assuming (iii), we obtain (ii) because the number of gaps has to be g.

Using Lemma 1.2 and Corollary 1.6, we are able to determine the gap sequence in case that $s_1, \ldots, s_\delta$ are collinear.

Checking case by case by use of Lemmas 1.2 and 1.4, we show a table of possible nongaps $N_{d,\delta}$ for $1 \leq \delta \leq 5$.

$N_{d,1} = N_{d,0} \cup \{(d-3)d+1\}$	general
$N_{d,2}^{(1)} = N_{d,1} \cup \{(d-4)d+2\}$	general
$N_{d,2}^{(2)} = N_{d,1} \cup \{(d-4)d+1\}$	s_1, s_2, P are collinear
$N_{d,3}^{(1)} = N_{d,2}^{(1)} \cup \{(d-4)d+1\}$	general
$N_{d,3}^{(2)} = N_{d,2}^{(1)} \cup \{(d-5)d+3\}$	s_1, s_2, s_3 are collinear but not P
$N_{d,3}^{(3)} = N_{d,2}^{(2)} \cup \{(d-5)d+1\}$	s_1, s_2, s_3, P are collinear
$N_{d,4}^{(1)} = N_{d,3}^{(1)} \cup \{(d-5)d+3\}$	general
$N_{d,4}^{(2)} = N_{d,3}^{(1)} \cup \{(d-5)d+2\}$	$s_1, \ldots, s_4$ general but $i(\gamma, \mathit{\Gamma}; P) = 2$ where γ is the conic passing through $s_1, \ldots, s_4, P$
$N_{d,4}^{(3)} = N_{d,3}^{(1)} \cup \{(d-5)d+1\}$	s_1, s_2, s_3, P are collinear but not s_4
$N_{d,4}^{(4)} = N_{d,3}^{(2)} \cup \{(d-6)d+4\}$	s_1, s_2, s_3, s_4 are collinear but not P
$N_{d,4}^{(5)} = N_{d,3}^{(3)} \cup \{(d-6)d+1\}$	s_1, s_2, s_3, s_4, P are collinear
$N_{d,5}^{(1)} = N_{d,4}^{(1)} \cup \{(d-5)d+2\}$	general
$N_{d,5}^{(2)} = N_{d,4}^{(1)} \cup \{(d-5)d+1\}$	$s_1, \ldots, s_5$ general but $\exists$ conic γ passing through $s_1, \ldots, s_5, P$ and $i(\gamma, \mathit{\Gamma}; P) = 1$
$N_{d,5}^{(3)} = N_{d,4}^{(2)} \cup \{(d-5)d+1\}$	$s_1, \ldots, s_5$ general but $\exists$ conic γ passing through $s_1, \ldots, s_5, P$ and $i(\gamma, \mathit{\Gamma}; P) = 2$
$N_{d,5}^{(4)} = N_{d,4}^{(1)} \cup \{(d-6)d+4\}$	$s_1, \ldots, s_4$ are collinear but not s_5, P
$N_{d,5}^{(5)} = N_{d,4}^{(3)} \cup \{(d-6)d+1\}$	$s_1, \ldots, s_4, P$ are collinear but not s_5
$N_{d,5}^{(6)} = N_{d,4}^{(4)} \cup \{(d-7)d+5\}$	$s_1, \ldots, s_5$ are collinear but not P
$N_{d,5}^{(7)} = N_{d,4}^{(5)} \cup \{(d-7)d+1\}$	$s_1, \ldots, s_5, P$ are collinear

2 General Case ($\delta \geq 2$)

Remember the definition of $N^{(1)}_{d,\delta}$, let $k = \min\{\ell \in \mathbf{N} | \delta \leq \dfrac{\ell(\ell+3)}{2}\}$. Then

$$N^{(1)}_{d,\delta} = N_d \cup \{n \in \mathbf{N} | n \geq (d-k-3)d + \frac{k(k+3)}{2} - \delta + 2\}.$$

In this section, we prove that for $(\mathit{\Gamma}; P)$ general, the semigroup of non-gaps of P is equal to $N^{(1)}_{d,\delta}$.

Let $\mathbf{P}_\ell \cong \mathbf{P}^{\ell(\ell+3)/2}$ be the linear system of divisors of degree ℓ on $\mathbf{P}^2$. Let

$$\mathbf{P}_\ell(s_1, \ldots, s_\delta) = \{\gamma \in \mathbf{P}_\ell | s_1, \ldots, s_\delta \in \gamma\},$$

and let

$$\mathbf{P}_k(s_1, \ldots, s_\delta; m) = \{\gamma \in \mathbf{P}_k(s_1, \ldots, s_\delta) | i(\mathit{\Gamma}, \gamma; P) \geq m\}.$$

Lemma 2.1 *Assume that*

$$(*) \qquad \begin{cases} \mathbf{P}_\ell(s_1, \ldots, s_\delta) = \emptyset & \text{if } \ell < k, \\ \mathbf{P}_k(s_1, \ldots, s_\delta; m) = \emptyset & \text{if } m > \dfrac{k(k+3)}{2} - \delta. \end{cases}$$

Then the Weierstrass gap sequence of $\mathit{\Gamma}$ at P is given by $\mathbf{N}^+ - N^{(1)}_{d,\delta}$.

Proof. By Lemma 1.2, every element of $N_{d,0}$ is a nongap. For $0 \leq n \leq d-3$ the natural number not belonging to $N_{d,0}$ are $nd+1, \ldots, nd+(d-n-2)$. Assume such a number $nd+\beta$ (hence $0 \leq n \leq d-3$, $1 \leq \beta \leq d-n-2$) is a gap. Then there exists a canonical adjoint curve γ of $\mathit{\Gamma}$ with

$$i(\gamma, \mathit{\Gamma}; P) = nd + \beta - 1.$$

Lemma 1.4 gives us that there exists $\gamma' \in \mathbf{P}_{d-3-n}(s_1, \ldots, s_\delta)$ with $i(\gamma', \mathit{\Gamma}; P) = \beta - 1$. But the hypothesis $(*)$ implies that this is impossible for $d-3-n < k$, i.e. $n > d-3-k$

or for $n = d-3-k$ and $\beta - 1 > \dfrac{k(k+3)}{2} - \delta$. So, the only possible gaps are

$$\begin{array}{llll}
1, \quad 2, & \dots & \dots, & d-2 \\
d+1,\ d+2, & \dots & \dots, & 2d-3 \\
2d+1, 3d+2, & \dots & \dots, & 3d-4 \\
 & \dots & \dots & \\
(d-4-k)d+1, & \dots & \dots, & (d-3-k)d-(d-2-k) \\
(d-3-k)d+1, & \dots & \dots, & (d-3-k)d+\dfrac{k(k+3)}{2}-\delta+1.
\end{array}$$

Since these are g mumbers, we obtain the gaps of C at P. It is clear that this set is $\mathbf{N}^+ - N^{(1)}_{d,\delta}$.

Theorem 2.2 *The hypothesis* $(*)$ *in Lemma* 2.1 *occurs.*

Proof. (Inspired by the proof of Proposition 3.1 in [8]). Take a union of d general lines in $\mathbf{P}^2$: $\mathit{\Gamma}_0 = L_1 \cup L_2 \cup \dots \cup L_d$.

Let $P_1 = L_1 \cap L_2$, $\{P_2, P_3\} = L_3 \cap (L_1 \cup L_2)$ and so on. Take $0 \le \delta \le \dfrac{(d-1)(d-2)}{2}$. The statement $(*)$ holds for $\mathit{\Gamma}_0$ instead of $\mathit{\Gamma}$ and $s_1 = P_1, \dots, s_\delta = P_\delta$ and P_0 suitably chosen on L_d.

Indeed, let $k = \min\{\ell \in \mathbf{N} | \delta \le \dfrac{\ell(\ell+3)}{2}\}$. Take $\ell < k$ and assume that $\gamma \in \mathbf{P}_\ell(P_1, \dots, P_\delta)$. Since

$$\{P_{\frac{(\ell+1)\ell}{2}+1}, \dots, P_{\frac{(\ell+2)(\ell+1)}{2}}\} = L_{\ell+2} \cap (L_1 \cup \dots \cup L_{\ell+1}) \subset \gamma$$

one has $\gamma = \gamma_{\ell-1} \cup L_{\ell+2}$ with $\gamma_{\ell-1} \in \mathbf{P}_{\ell-1}(P_1, \dots, P_{\frac{(\ell+1)\ell}{2}})$. Continuing this way one finds

$$\gamma = L_{\ell+2} \cup \gamma_{\ell-1} = L_{\ell+2} \cup L_{\ell+1} \cup \gamma_{\ell-2} = \dots = L_{\ell+2} \cup \dots \cup L_4 \cup \gamma_1,$$

where $\gamma_j \in \mathbf{P}_j(P_1, \dots, P_{\frac{(j+2)(j+1)}{2}}), (j = 1, \dots, \ell-1)$. Since P_1, P_2, P_3 are not collinear, this is impossible.

This already proves that $\mathbf{P}_\ell(P_1, \dots, P_\delta) = \emptyset$ for $\ell < k$. In particular $\mathbf{P}_k(P_1, \dots, P_{\frac{(k+1)(k+2)}{2}}) = \emptyset$. This implies $\dim(\mathbf{P}_k(P_1, \dots, P_\delta)) = \dfrac{k(k+3)}{2} - \delta$. Because $\delta \le \dfrac{(d-1)(d-2)}{2}$, $\{P_1, \dots, P_\delta\} \cap L_d = \emptyset$. So if some element of $\mathbf{P}_k(P_1, \dots, P_\delta)$ would contain L_d then $\mathbf{P}_{k-1}(P_1, \dots, P_\delta) \neq \emptyset$, a contradiction.

Hence, $\mathbf{P}_k(P_1, \ldots, P_\delta)$ induces a linear system of dimension $\frac{k(k+3)}{2} - \delta$ on L_d. For P_0 general on L_d and $\gamma \in \mathbf{P}_k(P_1, \ldots, P_\delta)$, this implies $i(\gamma, L_d; P_0) \leq \frac{k(k+3)}{2} - \delta$, hence

$$\mathbf{P}_k(P_1, \ldots, P_\delta; m) = \emptyset \quad \text{if } m > \frac{k(k+3)}{2} - \delta.$$

Claim: There exists a smooth (affine) curve T and $0 \in T$ and a family of plane curves of degree d

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook] \arrow[d, "p"'] & T \times \mathbf{P}^2 \arrow[dl, "p_T"] \\ T & \end{tikzcd}$$

with δ sections $S_1, \ldots, S_\delta : T \to \mathcal{C}$ satisfying the following properties:

1. $p^{-1}(0) = \Gamma_0 = L_1 \cup \cdots \cup L_d$:
2. $S_i(0) = P_i$ for $1 \leq i \leq \delta$:
3. for $r \in T - \{0\}$, $p^{-1}(r)$ is an irreducible curve, $S_i(r)$ is an ordinary node for $p^{-1}(r)$ and $p^{-1}(r)$ has no other singularities, P_0 is a total inflection point on $p^{-1}(r)$.

(For short, we call this a suited family of curves on $\mathbf{P}^2$ containing Γ_0 preserving the first δ nodes and the total inflection point P_0.)

Because of semi-continuity reasons it follows that for a general $r \in T$ the curve $p^{-1}(r)$ satisfies the statement $(*)$. So it is sufficient to prove the claim.

In order to prove the claim we start as follows. Let $\pi_1 : X_1 \to \mathbf{P}^2$ be the blowing-up of $\mathbf{P}^2$ at P_0. Let E_1 be the exceptional divisor and let $L_{d,1}$ be the proper transform of L_d. Let $P^{(1)} = L_{d,1} \cap E_1$. Blow-up X_1 at $P^{(1)}$ obtaining $\pi_2 : X_2 \to X_1$ with the exceptional divisor E_2 and let $L_{d,2}$ be the proper transform of $L_{d,1}$. Let $P^{(2)} = L_{d,2} \cap E_2$ and continue until one obtains

$$\pi : X = X_d \xrightarrow{\pi_d} X_{d-1} \xrightarrow{\pi_{d-1}} \cdots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} \mathbf{P}^2.$$

Write L_i for $\pi^{-1}(L_i)$ for $1 \leq i \leq d-1$ and let

$$\Gamma_0' = L_1 + \ldots + L_{d-1} + L_{d,d}.$$

For $1 \leq i \leq d-1$, let $\mu_i = \pi_{i+1} \circ \cdot \circ \pi_d$ and let L be a general line on $\mathbf{P}^2$. Then

$$\Gamma_0' \in \mathbf{P} := |d\pi^*(L) - \left(\sum_{i=1}^{d-1} \mu_i^*(E_i)\right) - E_d|$$

We are going to use a theorem of Tannenbaum [7, Theorem 2.13]. Since $L_{d,d}.K_X \geq 0$, we are not allowed to take $Y = \Gamma_0'$ on X in Tannenbaum's Theorem. Therefore we first prove the existence of an irreducible curve Γ_1' in $\mathbf{P}$ with enough nodes.

From Tannenbaum's Theorem it follows that there is a quasi-projective family $\mathbf{P}_d((d-1)(d-2)/2) \subset \mathbf{P}_d$ of dimension $\dfrac{d(d+3)}{2} - \dfrac{(d-1)(d-2)}{2}$ such that a general element belongs to a suited family of curves on $\mathbf{P}^2$ containing Γ_0 and preserving the first $\dfrac{(d-1)(d-2)}{2}$ nodes.

The condition $i(\gamma, L_d; P_0) \geq d$ for $\gamma \in \mathbf{P}_d((d-1)(d-2)/2)$ are at most d linear condition. Let

$$\mathbf{P}_d((d-1)(d-2)/2; d) = \{\gamma \in \mathbf{P}_d((d-1)(d-2)/2) | i(\gamma, L_d; P_0) \geq d\}.$$

One has $\Gamma_0 \in \mathbf{P}_d((d-1)(d-2)/2; d)$ and

$$\dim(\mathbf{P}_d((d-1)(d-2)/2; d)) \geq \frac{d(d+3)}{2} - \frac{(d-1)(d-2)}{2} - d = 2d - 1.$$

Let $\tilde{\mathbf{P}}$ be an irreducible component of $\mathbf{P}_d((d-1)(d-2)/2; d)$ containing Γ_0. Since Γ_0 is smooth at P_0, a general element of $\tilde{\mathbf{P}}$ is smooth at P_0. Let Γ_1 be a general element of $\tilde{\mathbf{P}}$. If Γ_1 is not irreducible then $i(\Gamma_1, L_d; P_0) = d$ implies that L_d is an irreducible component of Γ_1. Since $\{P_1, \ldots, P_{\frac{(d-1)(d-2)}{2}}\} \cap L_d = \emptyset$ also Γ_1 possesses $\dfrac{(d-1)(d-2)}{2}$ nodes none of them belonging to L_d. This implies $\Gamma_1 = L_d \cup \Gamma_2$, where Γ_2 belongs to a family of plane curves of degree $d-1$ on $\mathbf{P}^2$ containing $L_2 \cup \cdots \cup L_d$ and preserving the $\dfrac{(d-1)(d-2)}{2}$ nodes. Clearly, if a union of at least two of the lines $L_2, \ldots, L_d$ become irreducible in this deformation, some nodes have to disappear. Since this is not allowed, Γ_2 is the union of $d-1$ lines. But this would imply $\dim(\tilde{\mathbf{P}}) = 2d-2$, a contradiction. This proves that Γ_1 is irreducible.

Moreover Γ_1 belongs to a suited family of curves on $\mathbf{P}^2$ containing Γ_0 preserving the first $\dfrac{(d-1)(d-2)}{2}$ nodes and the total inflection point P_0. Because of semi-continuity, we can assume that $(*)$ holds for the first δ nodes of Γ_1.

Let Γ_1' be the proper transform of Γ_1 on X. Then $\Gamma_1' \in \mathbf{P}$ and we can apply Tannenbaum's Theorem to obtain a suited family of curves on X belonging to $\mathbf{P}$ containig Γ_1' and preserving the first δ nodes of Γ_1'. Projecting on $\mathbf{P}^2$ we obtain a suited family of curves on $\mathbf{P}^2$ containing Γ_1, preserving the first δ nodes of Γ_1 and the total inflection point P_0. This completes the proof of the claim.

Let

$$\mathbf{P}_d(d,\delta) = \left\{ \begin{array}{ll} \gamma \in \mathbf{P}_d : & \gamma \text{ is irreducible;} \\ & \gamma \text{ has a total inflection point and} \\ & \gamma \text{ has } \delta \text{ ordinary nodes and no other singularities} \end{array} \right\}.$$

Then Ran [6, The irreducibility Theorem (bis)] proves that $\mathbf{P}_d(d;\delta)$ is irreducible. This implies:

Theorem 2.3 *The normalization of a general nodal irreducible plane curve of degree d with δ nodes and possessing a total inflection point P has in general Weierstrass gap sequence given by $N^{(1)}_{d,\delta}$ at P.*

3 Case: Maximal Weight

Assume that $\delta \le d-2$ and remember the definition for $N^{(\max)}_{d,\delta}$ and $N^{(\max 2)}_{d,\delta}$ in the introduction.

Let P be a total inflection point on the nodal plane curve Γ of degree d with δ nodes, let $\alpha_1 < \cdots < \alpha_g$ be the Weierstrass gap sequence of P and let $N = \mathbf{N} - \{\alpha_1, \dots, \alpha_g\}$ be the semigroup of non-gaps of P.

Lemma 3.1 *For $1 \le i \le g$ one has $\alpha_i \le \alpha_i^{(\max)}$. Moreover if $N \ne N^{(\max)}_{d,\delta}$, then $\alpha_i \le \alpha_i^{(\max 2)}$ for $1 \le i \le g$.*

Proof. For $\delta \leq 2$ see §1, so assume that $\delta \geq 3$. Let $\alpha_{i,j} = (d-i-2)d+j$, $1 \leq j \leq i \leq d-2$. They are just the members of $\mathbf{N} - N_d$.

Since $N_d \subset N$, by Lemma 1.2, N is the union of N_d and δ values of $\alpha_{i,j}$. Moreover, if $\alpha \in N$ then $\{\alpha+d-1, \alpha+d\} \subset N$. So, if the number of values $\alpha_{i',j}$ belonging to N with $i' < i$ is less than δ, then $\alpha_{i,j_0} \in N$ for some $1 \leq j_0 \leq i$. Each of $N_{d,\delta}^{(\max)}$ and $N_{d,\delta}^{(\max2)}$ does not possess two values $\alpha_{i,j_1} \neq \alpha_{i,j_2}$ for each i. Hence, if $\{\alpha_{2,1}, \alpha_{2,2}\} \subset N$, then

$$^{\#}\{\alpha_{i',j'} \in N | i' < i, j' \geq j\} \geq \ ^{\#}\{\alpha_{i',j'} \in N_{d,\delta}^{(\max2)} | i' < i, j' \geq j\} \quad \text{for } \forall i, j.$$

So, we have $\alpha_k \leq \alpha_k^{(\max2)}$ for $1 \leq k \leq g$. In particular, $\alpha_k \leq \alpha_k^{(\max)}$. But if $\{\alpha_{2,1}, \alpha_{2,2}\} \not\subset N$, then $N \in \{N_{d,\delta}^{(\max)}, N_{d,\delta}^{(\max2)}\}$ because of Corollary 1.6.

This completes the proof of the lemma.

Proposition 3.2 *If $d \geq 2\delta+1$, then $N_{d,\delta}^{(\max)}$ occurs as the semigroup of the non-gaps of a total inflection point and if $d \geq 2\delta$, then so does $N_{d,\delta}^{(\max2)}$.*

Proof. Fix $\delta+1$ points $P, P_1, \ldots, P_\delta$ on an arbitrary line L. For $i = 1, \ldots, \delta$, take general lines L_i and L_i' passing through P_i. Let T be a general line passing through P and let C be a curve of degree $d-2\delta-1$ which does not pass through any one of $P, P_1, \ldots, P_\delta$ and the common point of each pair of the above curves. Let

$$\begin{aligned} C_1 &= dL \\ C_2 &= T + C + L_1 + L_1' + \cdots + L_\delta + L_\delta'. \end{aligned}$$

Let $\mathbf{P}$ be the pencil generated by C_1 and C_2. By Bertini's theorem, a general element Γ of $\mathbf{P}$ is a curve of degree d with δ ordinary nodes at $P_1, \ldots, P_\delta$ as its only singularities and P is a total inflection point of Γ with tangent line T. In particular, if Γ would not be irreducible then $\Gamma = T + \Gamma'$. But then T would be a fixed component of $\mathbf{P}$, which is not true. Hence Γ is irreducible. Because of Corollary 1.6, the semigroup of nongaps of P is $N_{d,\delta}^{(\max)}$.

Next, we prove the latter part. Fix δ points $P_1, \ldots, P_\delta$ on an arbitrary line L and a point P not on L. For $i = 1, \ldots, \delta$, let L_i be the line joining P and P_i and let L_i' be general lines passing through P_i. Let T and T' be general lines passing through P but not

any of P_i and let C be a curve of degree $d-\delta-2$ which does not pass through any one of $P, P_1, \ldots, P_\delta$ and the common point of each pair of the above curves. Let

$$\begin{aligned} C_1 &= 2(L_1+\cdots+L_\delta)+(d-2\delta)T' \\ C_2 &= L+T+C+L'_1+\cdots+L'_\delta. \end{aligned}$$

Let $\mathbf{P}$ be the pencil generated by C_1 and C_2. Again, by Bertini's theorem, a general element Γ of $\mathbf{P}$ is a curve of degree d with δ ordinary nodes at $P_1, \ldots, P_\delta$ as its only singularities and P is a total inflection point of Γ with tangent line T. Also Γ is irreducible, by Corollary 1.6, the semigroup of nongaps of P is $N_{d,\delta}^{(\max 2)}$.

Remark 3.3. Define $N_{d,3}^{(\max 3)} = N_{d,3}^{(\max 4)} = N_{d,3}^{(1)}$ and for $\delta > 3$ we define inductively $N_{d,\delta}^{(\max 3)} = N_{d,\delta-1}^{(\max 3)} \cup \{(d-\delta-1)d+1\}$ and $N_{d,\delta}^{(\max 4)} = N_{d,\delta-1}^{(\max 4)} \cup \{(d-\delta-1)d+\delta-1\}$. As above one can check that, for $\delta \geq 3$ and $N \notin \{N_{d,\delta}^{(\max)}, N_{d,\delta}^{(\max 2)}\}$ one has $\alpha_k \leq \alpha_k^{(\max 3)}$ for $1 \leq k \leq g$ and, for $\delta \geq 5$ and $N \notin \{N_{d,\delta}^{(\max)}, N_{d,\delta}^{(\max 2)}, N_{d,\delta}^{(\max 3)}\}$ one has $\alpha_k \leq \alpha_k^{(\max 4)}$ for $1 \leq k \leq g$. Moreover $N_{d,\delta}^{(\max 3)}$ (resp. $N_{d,\delta}^{(\max 4)}$) occurs if and only if exactly $\delta-1$ nodes are on a line L_0 and $P \in L_0$ (resp. $P \notin L_0$). As above one can also discuss the existence.

If one wants to continue, then one has to start making an analysis of the case where the nodes are on a conic. Another direction of further investigation could be: let $3 \leq \delta' \leq \frac{d}{2}$, what is the general situation for N if δ' nodes are on a line ? Probably reasoning as in §2, one obtains an answer.

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